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# Conformal invariance in general relativity

G V Bicknell

Department of Applied Mathematics, University of Sydney, Sydney 2006, New South Wales, Australia

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**Abstract.** Bramson recently showed how the interaction of the gravitational field with a massive spinor field could be cast into a conformally invariant form by the addition of a scalar field which is constant in a family of canonical gauges. Here Bramson's treatment is extended to include the interaction of gravitation with matter in general and it is shown that the dependence of particle masses on the scalar field $\phi$, $m = m_0\phi$, assumed by Bramson, is in fact a necessary one. In addition, the equations of motion of test particle in a general gauge are derived.

## 1. Introduction

Bramson (1974) showed that the equations governing the interaction of the gravitational field with a massive spinor field could be cast into a form which is invariant under the conformal transformation

$$g_{ij} = e^{\sigma} g'_{ij}, \tag{1}$$

where $\sigma$ is an arbitrary space–time function. Central to this result was the introduction of a conformally invariant scalar field which transforms under equation (1) according to

$$\phi = \phi' \, e^{-\sigma/2}, \tag{2}$$

and the assumption that particle masses are of the form

$$m = m_0\phi. \tag{3}$$

In the particular gauge in which $\phi$ is constant, Bramson's equations reduce to the Einstein form. The purpose of this paper is to show the necessity for equation (3) for matter in general.

## 2. Conformally invariant Lagrangian

To represent the general case of gravitation interacting with matter we adopt the action

$$I = \int (\tfrac{1}{12}\phi^2 R - \tfrac{1}{2}\phi_{,k}\phi^{,k} + \tfrac{1}{4}\alpha\phi^4 - 8\pi L_m(\psi^A, h_{(a)i}, \phi))\sqrt{-g}\, d^4x, \tag{4}$$

the free field terms being those already used by Bramson. The last term in the above action is a general matter term depending on matter fields $\psi^A$, a tetrad field (necessitated by the inclusion of spinor variables), and the scalar field $\phi$. The metric and tetrad fields are related by

$$g_{ij} = h_{(a)i} h^{(a)}{}_j. \tag{5}$$

The reader is referred to Weinberg (1972, p 370) for a description of the tetrad formalism. Variation of $I$ with respect to $h_{(a)i}$ and $\phi$ with the usual conditions on the boundary of integration gives the field equations

$$\tfrac{1}{12}\phi^2 G_{ij} + \tfrac{1}{3}\phi_{,i}\phi_{,j} - \tfrac{1}{12}\phi_{,k}\phi^{,k} g_{ij} + \tfrac{1}{6}\phi(\Box\phi g_{ij} - \phi_{;ij}) + \tfrac{1}{8}\alpha\phi^4 g_{ij} = -8\pi T_{ij}, \tag{6}$$

$$\Box\phi + \tfrac{1}{6}R\phi + \alpha\phi^3 = -8\pi S, \tag{7}$$

where the stress energy tensor $T_{ij}$ and the source term for the scalar field are defined by

$$\delta \int L_m \sqrt{-g}\, d^4x = \int (2T^{i(a)}\delta h_{(a)i} + S\delta\phi)\sqrt{-g}\, d^4x. \tag{8}$$

It is presumed that in the limit of a large number of interacting and non-interacting fields, $T^{ij}$ and $S$ become the corresponding quantities for matter in bulk.

## 3. Coordinate and conformal invariance identities

The free field integral in equation (4) is invariant under coordinate and conformal transformations. Thus, for the theory derived from equation (4) to be covariant and conformally invariant, the same is demanded of the matter part of the action $\int L_m\sqrt{-g}\, d^4x$. This leads to the two sets of identities

$$T^{ij}{}_{;j} = \tfrac{1}{2}S\phi^{,i}, \tag{9}$$

$$T = \tfrac{1}{2}S\phi. \tag{10}$$

## 4. Equations of motion of a test particle

The above identities may be used to derive the equations of motion of a test particle and the dependence of mass upon $\phi$. We initially consider a body of finite extent tracing out a world tube in space–time. A test particle is taken to be the limiting case of the world tube contracting to a world line. We represent the body by a concentration of energy with negligible stress, so that

$$T_{ij} = \rho c^2 V_i V_j, \tag{11}$$

where $\rho$ is the density of matter within the world tube and $V^i$ its four velocity.

Substitution of equation (11) into the coordinate identities (9) and also using equation (10) gives

$$\rho c^2 V^i{}_{;j} V^j + (\rho c^2 V^j)_{;j} V^i = -\rho c^2(\phi^{,i}/\phi). \tag{12}$$

Transvection with $V^i$ gives

$$(\rho c^2 V^j)_{;j} = \rho c^2 \frac{\phi_{,j}}{\phi} V^j. \tag{13}$$

Consider a section of the world tube capped by two three-spaces $V_3$ and $V_3'$. They are constructed so that they are normal to the limiting world line at the points P and P′. Integration of equation (13) throughout the world tube, converting the integral of the divergence to a surface integral leads to

$$-\int_{V_3'} \rho c^2 V_j N^j \, d_3 v + \int_{V_3} \rho c^2 V_j N^j \, d_3 v = \int_{V_4} \rho c^2 \frac{\phi_{,j}}{\phi} V^j \, d_4 v, \tag{14}$$

(cf Synge 1960, p 46).

$N^j$ is the normal to the three-spaces $V_3$ and $V_3'$. There is no contribution to the left hand side of equation (14) from the sides of the world tube since these are parallel to $V^i$. As the radius of the world tube shrinks to zero, the difference between $N^j$ and $V^j$ becomes insignificant. ($N^i$ cannot be assumed to be parallel to $V^i$ throughout the world tube, for that would imply a restriction on $V^i$.) Therefore, for an arbitrarily thin world tube

$$\int_{V_3'} \rho c^2 \, d_3 v - \int_{V_3} \rho c^2 \, d_3 v = \int_{V_4} \rho c^2 \frac{\phi_{,j}}{\phi} V^j \, d_4 v. \tag{15}$$

The integrals on the left are merely the energies of the test particle at the points P′ and P, respectively. Thus

$$mc^2(\mathrm{P}') - mc^2(\mathrm{P}) = \int_{V_4} \rho c^2 \frac{\phi_{,j}}{\phi} V^j \, d_4 v. \tag{16}$$

Now let the points P and P′ be separated by a proper distance d$s$. The element of the four-volume sandwiched between $V_3$ and $V_3'$ may now be written as $d_3 v \, ds$ where $d_3 v$ is the element of three-volume of $V_3$ and equation (16) becomes

$$d(mc^2) = ds \int_{V_3} \rho c^2 \frac{\phi_{,j}}{\phi} V^j \, d_3 v. \tag{17}$$

Ignoring the small contribution to $\phi$ from the test particle and the difference between $V_j$ and its world line value we have

$$d(mc^2) = ds \frac{\phi_{,j}}{\phi} V^j \int_{V_3} \rho c^2 \, d_3 v,$$

so that

$$\frac{d}{ds}(mc^2) = mc^2 \frac{\phi_{,j}}{\phi} V^j, \tag{18}$$

or

$$\frac{dm}{ds} = \frac{mc}{\phi} \frac{d\phi}{ds}. \tag{19}$$

Equation (19) immediately implies that

$$m = m_0 \phi. \tag{20}$$

Therefore it has been shown that conformal invariance requires the dependence of mass upon $\phi$ assumed by Bramson.

This treatment of the equations of motion may be completed by substituting equation (13) into equation (12) and multiplying by $m/\rho$ to obtain

$$mc^2 V^i{}_{;j} V^j + mc^2 \frac{\phi_{,j}}{\phi} V^j V^i = -mc^2 \frac{\phi^{,i}}{\phi}. \tag{21}$$

Using equation (18) this equation may be written as

$$\frac{\delta}{\delta s}(mc^2 V^i) = -mc^2 \frac{\phi^{,i}}{\phi}. \tag{22}$$

## 5. Conclusions

We have extended here Bramson's treatment of conformally invariant general relativity. It has been shown that the relation between mass and the scalar field, $m = m_0\phi$, is a necessary one in a general gauge. It has also been shown that this conformal invariance should be accompanied by a generalization of the equations of motion to equations (22). Note that in the gauge in which $\phi$ is constant, particle masses are constant, the equations of motion are geodesics, stress energy is conserved and the field equations reduce to Einstein's with a cosmological term which may or may not be zero.

One may ask whether the role of the scalar field $\phi$ as described here is to be the only role of a conformally invariant scalar field in general relativity. This matter will be dealt with in forthcoming publications. The work described in this paper also forms the starting point for conformally invariant generalizations of general relativity. This will also be considered in future papers.

## References

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